Roberge-Weiss Transitions

We want to investigate the confinement / de-confinement transition in SU(N) YM theory.

As an order-parameter, we introduce the Polyakov line:

$$L = \frac{1}{N} \operatorname{tr} P \exp\left\{ i \int_0^\beta d\tau \, A_\mu \dot{x}^\mu \right\}$$

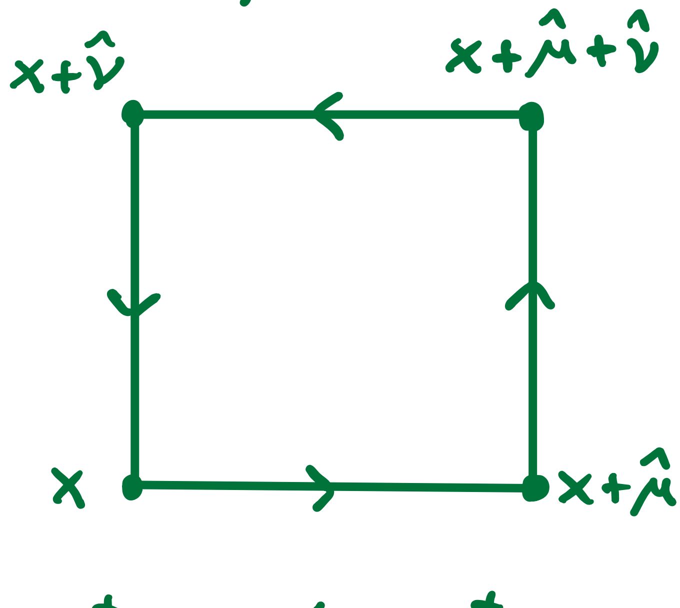
This is related to the Free Energy in the presence of a static quark source:

$$F = -\frac{1}{\beta} \log \langle L \rangle$$

part I: pure gauge

SU(N) YM, without fermions, exibits a Z_n symmetry. This is most easily understood in terms of LQCD and the Wilson-action:

$$S_W = -\frac{1}{g^2}\sum_{\square}\left(W_\square + W_\square^\dagger\right)$$



with

$$W_\square = \mathrm{tr}\, U_\mu(x)\, U_\nu(x+\hat{\mu})\, U_\mu^\dagger(x+\hat{\nu})\, U_\nu^\dagger(x)$$

Given the above action, we can express $\langle L \rangle$ in the following way:

$$\langle L \rangle = \frac{1}{Z} \int DU \, L \, e^{-S_W[U]}$$

Now, consider the symmetry-transformation:

$$U_\mu(x_0) \longmapsto \underline{z} \, U_\mu(x_0)$$

Both the Wilson action and Haar-measure are invariant under the Z_n transform, whereas $L \longmapsto z_k L$, since $z_k = e^{2\pi i k/N}$, $k = 1, \ldots, N$, we find:

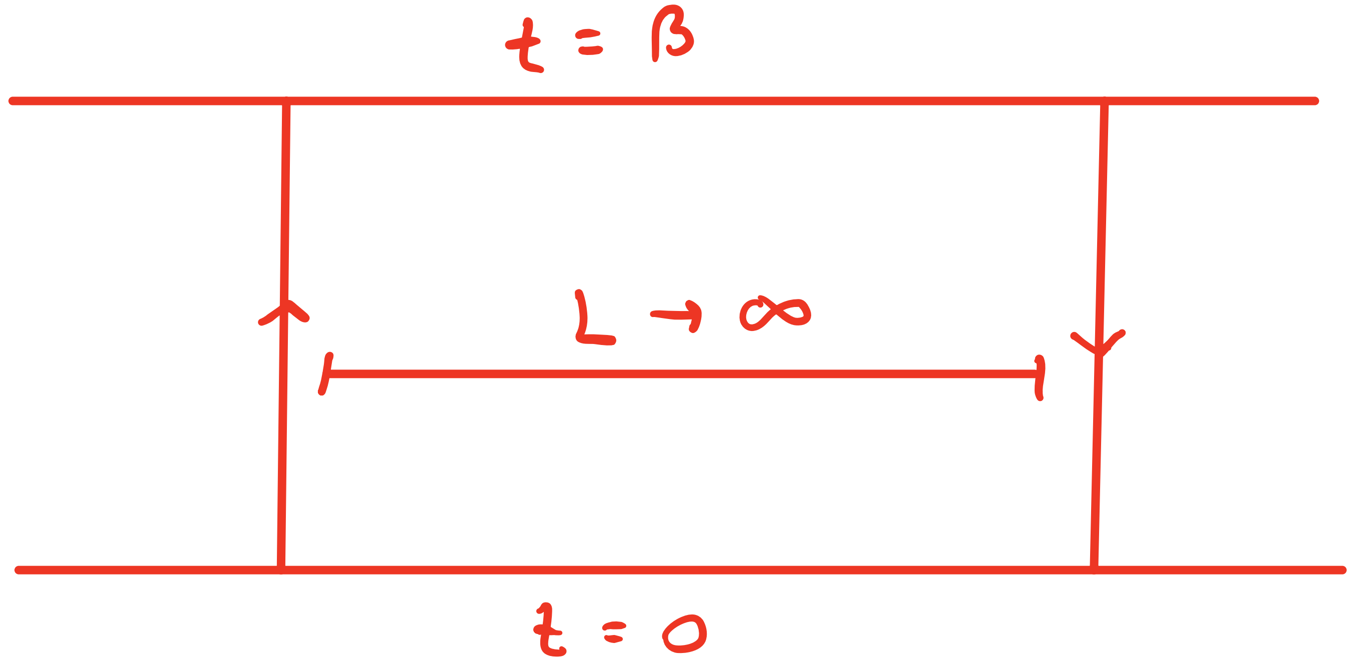
$$\langle L \rangle = z_k \langle L \rangle \quad \Rightarrow \quad \boxed{\langle L \rangle = 0}$$

This implies that the free-energy is ∞, which in turn implies confinement!

(The setup with one static quark source, can be regarded as the limit of a static quark anti-quark pair, for which:

$$F = -\frac{1}{\beta}\log\langle L^2\rangle \approx -\frac{1}{\beta}\log\langle L\rangle^2$$

$$\sim -\frac{1}{\beta}\log\langle L\rangle$$ cluster-decomposition



part II: adding frmions

when frmions are introduced, the Z_n symmetry is broken.

This is most easily seen from the descrete frmion action:

$$S_F = a^4 \sum_{x,\mu} i\, \bar{\psi}(x) \gamma^\mu \frac{U_\mu(x)\psi(x+\hat{\mu}) - \psi(x)}{a}$$

masses are trivial to add

Clearly, this action is NOT invariant under:

$$U_\mu(x_0) \longmapsto \underline{\underline{z}}\, U_\mu(x_0)$$

However, it turns out that an analogous "symmetry" remains!

Part III : imaginary chemical potential

We start by introducing an imaginary quark chemical potential Γ, in the presence of which the partition function Z takes the form:

$$Z = \mathrm{Tr}\, e^{-\beta \hat{H} + i \Gamma \hat{N}}$$

with

$$\hat{N} = \int d^3x\, \psi^{\dagger} \psi$$

We now consider 2 cases for the contents of the spectrum of our theory:

Case 1: free quarks

For this case, all eigenstates of $\hat{N}$ obeys:

$$\hat{N} |q\rangle = n_q |q\rangle , \quad n_q \in \mathbb{Z}$$

Thus, $Z(\Gamma + 2\pi) = Z(\Gamma)$.

Case 2: only singlets

For this case, all eignstates of $\hat{N}$ obeys:

$$\hat{N} |s\rangle = N n_s |s\rangle , \quad n_s \in \mathbb{Z}$$

Thus, $Z\left(\Gamma + \frac{2\pi}{N}\right) = Z(\Gamma)$

It seems that the periodicity of $Z(\Gamma)$ "knows" about confinment !

It turns out that one can show:

$$Z(\Gamma) = Z\left(\Gamma + \frac{2\pi}{N}\right)$$

even at high T. This suggests that, also at high T, it is only possible to excite color-singlet states!

Regardless, we now look for phase-transitions in the (Γ, T) plane by use of:

- perturbation theory, at high T

- strong coupling expansion, at low T

Part VI: The perturbative regime

We start out by transforming to a gauge in which:

- A_0 is τ-independent
- A_0 is diagonal with:

$$\phi \equiv g A_0 = \begin{pmatrix} \phi_1 & & \\ & \ddots & \\ & & \phi_N \end{pmatrix}$$

Because L must be unitary, we must require:

$$\sum_\alpha \phi_\alpha = 0 \mod 2\pi/\beta$$

$$L = \frac{1}{N} \operatorname{tr} e^{i\beta\phi}$$

Actually, in the absence of fermions, $A_i = 0$ and ϕ_α with:

$$\sum_\alpha \phi_\alpha = 0 \mod 2\pi/\beta$$

constitute classical solutions of the system (minima of action), and so it make sense to expand around these in a perturbative framework.

when this is done, one find that, to 1-loop order:

$$V_{eff}^{g}(\{\phi_\alpha\}) = \frac{1}{24}\pi^2 T^4 \times \sum_{\alpha,\beta}^{N} \left[1 - \left(\left[\frac{\beta\phi_\alpha}{\pi} - \frac{\beta\phi_\beta}{\pi}\right]_{\text{mod } 2} - 1\right)^2\right]^2$$

It is not difficult to see,

that V_{eff}^{g} has a minimum at:

$\phi_\alpha = 0$ leading to $V_{eff}^{g} = \dfrac{\pi^2 T^4 N^2}{24}$

Exactly the same is true for:

$$\phi_\alpha = \frac{2\pi k}{\beta N}$$

This is to be expected, since

shifting $\phi_\alpha \longrightarrow \phi_\alpha + \dfrac{2\pi k}{\beta N}$

exactly corresponds to

performing a Z_n transform on L

If we now introduce fermions

to the system, the degeneracy

of the L minima will be lifted,

and only: $\phi_\alpha = 0$

remains as a true global minimum.

Expanding around this minimum and computing $\langle L \rangle$ to 1-loop order, one finds that:

$$V_{eff}^{f}(\{\phi_\alpha\}) = -\frac{1}{12}\pi T^4 \sum_\alpha \left[1 - \left(\left[\frac{\beta\phi_\alpha}{\pi} - 1\right]^2 - 1\right)^2\right]^2$$

Non-zero chemical potential:

The effect of introducing μ is effectively to set:

$$A_0 \longrightarrow A_0 + \frac{\mu}{\beta g}\mathbb{1}_N$$

Seems to be a missing $-i$ somewhere... NO!!

This changes V^f_{eff} but not V^g_{eff}:

$$V^g_{eff}(\{\phi_\alpha + \frac{\Gamma}{\beta}\}) = V^g_{eff}(\{\phi_\alpha\})$$

Thus, for $-\pi < \Gamma < +\pi$, and around $\phi_\alpha = 0$, we find that:

$$V_{eff}(\{\phi_\alpha = 0\}, \Gamma)$$
$$= -\frac{1}{12}\pi^2 T^4 \left[1 - \left(\frac{\Gamma}{\pi}\right)^2\right]^2$$

From $Z(\Gamma) = Z(\Gamma + \frac{2\pi k}{N})$, it follows that:

$$V_{eff}(\{\phi_\alpha = \frac{2\pi k}{\beta N}\}, \Gamma)$$

$$= V_{eff}(\{\phi_\alpha = 0\}, \Gamma - \frac{2\pi k}{N})$$

"Draw V_{eff} figure"

Since the free energy $F(\Gamma)$, is the minimum over all $V_{eff}(\{\phi_\alpha = 0\}, \Gamma)$, we see that the free energy has "cusps" at $\Gamma = \frac{2\pi(k+\frac{1}{2})}{N}$, and thus, the system will have several 1st order phase-transitions at these values of Γ.

Part V: The non-perturbative regime

In the "hopping"-parameter expansion:

$$Z = \int_{PBC} DU \exp\left\{ \frac{1}{g^2} \sum \square + \sum \bigcirc \right\}$$

$$K = (2ma+8)^{-1}$$

one finds that:

$$\langle L(\Gamma) \rangle \propto K^{N_t} e^{i\Gamma}$$

which is clearly an analytic function of Γ, which implies that the free energy is free of any "cusps".

Lattice results for SU(2):

- N. Weiss, 1985

Lattice results for SU(3):

- G. Endrödi, A. Chabane, 2021

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